Lecture 05
Defuzzification
解模糊
Defuzzification means the **fuzzy-to-crisp** conversions.

- The fuzzy results generated cannot be used to the applications.
- It is necessary to convert the fuzzy quantities into crisp quantities for further processing.

Defuzzification has the capability to

- reduce a fuzzy set to a crisp single-valued quantity or as a set.
Lambda Cuts
- for Fuzzy Sets
- for Fuzzy Relations

Defuzzification Methods
- Maxima method
- Centroid method
- Weighted average method
- Middle-of-maxima method
- First-of-maxima or last-of-maxima
Consider a fuzzy set $A$

- Its **lambda cut** denoted by $A_\lambda$ ($0 \leq \lambda \leq 1$)
- $A_\lambda$ is a **crisp set**:

$$A_\lambda = \left\{ x / \mu_A(x) \geq \lambda \right\},$$

- the value of lambda cut set is $x$, when the membership value corresponding to $x$ is greater than or equal to the specified $\lambda$.  

- This lambda cut set can also be called as **alpha cut set**.
Properties of Lambda Cut Sets

(1) \( \left( A \cup B \right)_\lambda = A_\lambda \cup B_\lambda \)
(2) \( \left( A \cap B \right)_\lambda = A_\lambda \cap B_\lambda \)
(3) \( \left( \overline{A} \right)_\lambda \neq (A_\lambda) \) except for a value of \( \lambda = 0.5 \)
(4) For any \( \lambda < \alpha \), where \( \alpha \) varies between 0 and 1, it is true that, \( A_\alpha \subset A_\lambda \), where the value of \( A_0 \) will be the universe defined.

The standard set of operations on fuzzy sets is similar to the standard set operations on lambda cut sets.
Considering a fuzzy relation $R$. A fuzzy relation can be converted into a crisp relation by depending the lambda cut relation of the fuzzy relation as:

$$R_{\lambda} = \{x, y / \mu_R (x, y) \geq \lambda\}.$$

Properties of Lambda Cut Relations

1. $\left( R \cup S \right)_{\lambda} = R_{\lambda} \cup S_{\lambda}$.
2. $\left( R \cap S \right)_{\lambda} = R_{\lambda} \cap S_{\lambda}$.
3. $\left( \overline{R} \right)_{\lambda} \neq (\overline{R_{\lambda}})$.
4. For $\lambda \leq \alpha$, where $\alpha$ between 0 and 1, then $R_{\alpha} \subseteq R_{\lambda}$. 
Example 1. Two fuzzy sets $P$ and $Q$ are defined on $x$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.7</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>$Q$</td>
<td>0.9</td>
<td>0.6</td>
<td>0.3</td>
<td>0.2</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Find the following $\lambda$ cut sets:

(a) $\left( \overline{P} \right)_{0.2}$  
(b) $(Q)_{0.3}$  
(c) $\left( P \cup Q \right)_{0.5}$  
(d) $\left( P \cap Q \right)_{0.4}$  
(e) $\left( Q \cup \overline{P} \right)_{0.8}$  
(f) $\left( P \cup \overline{P} \right)_{0.2}$
Lambda Cuts: Examples

**Solution.** Given

\[ P = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}, \]

\[ Q = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5} \right\}. \]

We have

\[ \overline{P} = \]

\[ \overline{Q} = \]

Then we can calculate

(a) \( \left( \overline{P} \right)_{0.2} = \)

(b) \( \left( \overline{Q} \right)_{0.3} = \)
Lambda Cuts: Examples

**Solution. Given**

\[ P = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}, \]

\[ Q = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} \right\}. \]

We have

\[ \overline{P} = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}, \]

\[ \overline{Q} = \left\{ \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} \right\}. \]

Then we can calculate

\[ (c) \left( P \cup Q \right) = \left( P \cup Q \right)_{0.6}. \]
Solution. Given

\[ P = \left\{ \begin{array}{c}
0.1 \\
0.2 \\
0.7 \\
0.5 \\
0.4
\end{array} \right\} \left\{ \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{array} \right\} \]

\[ Q = \left\{ \begin{array}{c}
0.9 \\
0.6 \\
0.3 \\
0.2 \\
0.8
\end{array} \right\} \left\{ \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{array} \right\} \]

We have

\[ \overline{P} = \left\{ \begin{array}{c}
0.9 \\
0.8 \\
0.3 \\
0.5 \\
0.6
\end{array} \right\} \left\{ \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{array} \right\} \]

\[ \overline{Q} = \left\{ \begin{array}{c}
0.1 \\
0.4 \\
0.7 \\
0.8 \\
0.2
\end{array} \right\} \left\{ \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{array} \right\} \]

Then we can calculate

\[ (d) \left( P \cup \overline{P} \right) = \left( P \cup \overline{P} \right)_{0.8} \]
**Solution.** Given

\[ P = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}, \]

\[ Q = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} \right\}. \]

We have

\[ \overline{P} = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}, \]

\[ \overline{Q} = \left\{ \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} \right\}. \]

Then we can calculate

\[ \begin{pmatrix} e \end{pmatrix} \left( P \cap Q \right) = \begin{pmatrix} 0.4 \end{pmatrix}. \]
Solution. Given

\[ P = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}, \]

\[ Q = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} \right\}. \]

We have

\[ \overline{P} = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}, \]

\[ \overline{Q} = \left\{ \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} \right\}. \]

Then we can calculate

\[ (f) \left( \frac{\overline{P} \cap \overline{P}}{\overline{P} \cap \overline{P}} \right) = \left( \frac{\overline{P} \cap \overline{P}}{0.8} \right). \]
Example 2. The fuzzy sets $A$ and $B$ are defined in the universe $X = \{0, 1, 2, 3\}$, with the following membership functions:

\[
\mu_A(x) = \frac{2}{x+3}, \\
\mu_B(x) = \frac{2x}{x+5}.
\]

Define the intervals along $x$-axis corresponding to the $\lambda$ cut sets for fuzzy sets $A$ and $B$ for following values of $\lambda$: $\lambda = 0.2, 0.5, 0.6$. 
Solution. The membership degrees for each elements:

\[
\begin{array}{cccccc}
X & 0 & 1 & 2 & 3 \\
\mu_A(x) = 2/(x+3) & 2/3 & 2/4 & 2/5 & 2/6 \\
\mu_B(x) = 2x/(x+5) & 0 & 1/3 & 4/7 & 3/4 \\
\end{array}
\]

That is,

\[
\begin{array}{cccccc}
X & 0 & 1 & 2 & 3 \\
\mu_A(x) & 0.67 & 0.5 & 0.4 & 0.33 \\
\mu_B(x) & 0 & 0.33 & 0.57 & 0.75 \\
\end{array}
\]

Then

(a) When \( \lambda = 0.2 \)
\[
\begin{align*}
A_{0.2} &= \ \\
B_{0.2} &= \\
\end{align*}
\]

(b) When \( \lambda = 0.5 \)
\[
\begin{align*}
A_{0.5} &= \ \\
B_{0.5} &= \\
\end{align*}
\]

(c) When \( \lambda = 0.6 \)
\[
\begin{align*}
A_{0.6} &= \ \\
B_{0.6} &= \\
\end{align*}
\]
Example 3. For the fuzzy relation:

\[
R = \begin{bmatrix}
1 & 0.2 & 0.3 \\
0.5 & 0.9 & 0.6 \\
0.4 & 0.8 & 0.7 \\
\end{bmatrix},
\]

find the \( \lambda \) cut relations for the following values of \( \lambda = 0+, 0.2, 0.9, 0.5. \)
Solution. Given

\[ R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}, \]

we have

\[ R_{0.0} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]
Outline

- Lambda Cuts
  - for Fuzzy Sets
  - for Fuzzy Relations

- Defuzzification Methods
  - Maxima method
  - Centroid method
  - Weighted average method
  - Middle-of-maxima method
  - First-of-maxima or last-of-maxima
The output of an entire fuzzy process can be union of two or more fuzzy sets.
Definition 1. Defuzzification is a process to select a representative element from the fuzzy output inferred from the fuzzy rule-based system.

There are various methods used for defuzzifying the fuzzy output functions:
- Maxima method
- Centroid method
- Weighted average method
- Middle-of-maxima method
- First-of-maxima or last-of-maxima
- ...
(1) Maxima

\[ \mu_{C_\infty}(z^*) \geq \mu_{C_\infty}(z), \quad \text{for all } z \in Z \]
(2) Centroid method

\[ z^* = \frac{\int \mu_C(z) zdz}{\int \mu_C(z) dz}, \quad \text{for all } z \in Z \]
(3) Weighted average method

- This method can be used only for symmetrical output membership functions
- Weighting each membership function in the obtained output by its largest membership value

\[ z^* = \frac{\sum \mu_C(z) \, z}{\sum \mu_C(z)} \]

\[ z^* = \frac{a \, (0.8) + b \, (0.6)}{0.8 + 0.6} \]
(4) **Middle-of-maxima method**

- The present of the maximum membership need not be unique,
  - i.e., the maximum membership need not be a single point, it can be a range.

\[ z^* = \frac{a + b}{2}, \]
Defuzzification Methods

(5) First-of-Maxima

\[ z_0 = \min \{ z \mid C(z) = \max_w C(w) \} . \]

Last-of-Maxima

\[ z_0 = \max \{ z \mid C(z) = \max_w C(w) \} . \]
Example 4. For the given membership function as shown, determine the defuzzified output value using different defuzzification methods.
Defuzzification: Examples

\[ \mu(z) = \begin{cases} 
0.35z & 0 \leq z < 2 \\
0.7 & 2 \leq z < 2.7 \\
z - 2 & 2.7 \leq z < 3 \\
1 & 3 \leq z < 4 \\
-0.5z + 3 & 4 \leq z \leq 6 
\end{cases} \]
Defuzzification: Examples

(1) **Maxima method**

Not applicable since there is no a single maximum point.

(2) **Centroid method**

\[
\mu(z) = \begin{cases} 
0.35z & 0 \leq z < 2 \\
0.7 & 2 \leq z < 2.7 \\
z-2 & 2.7 \leq z < 3 \\
1 & 3 \leq z < 4 \\
-0.5z + 3 & 4 \leq z \leq 6 
\end{cases}
\]

\[
z^* = \frac{\int \mu_C(z) zdz}{\int \mu_C(z) dz}
\]

Numerator = \[\int_0^2 0.35 z^2 \, dz + \int_2^{2.7} 0.7 z \, dz + \int_2^{3} (z-2) \, dz + \int_3^4 z \, dz + \int_4^6 (-0.5z^2 + 3z) \, dz\]

\[= 10.98.\]

Denominator = \[\int_0^2 0.35 z \, dz + \int_2^{2.7} 0.7 \, dz + \int_2^{3} (z-2) \, dz + \int_3^4 z \, dz + \int_4^6 (-0.5z + 3) \, dz\]

\[= 3.445.\]

\[
z^* = \frac{\text{Numerator}}{\text{Denominator}} = \frac{10.98}{3.445} = 3.187.
\]
Defuzzification: Examples

(3) Weighted average method

Not applicable since the membership functions are not symmetrical.

(4) Middle-of-maxima method

\[ z^* = \frac{3 + 4}{2} = 3.5 \]

\[
\mu(x) = \begin{cases} 
0.35x & 0 \leq x < 2 \\
0.7 & 2 \leq x < 2.7 \\
x - 2 & 2.7 \leq x < 3 \\
1 & 3 \leq x < 4 \\
-0.5x + 3 & 4 \leq x \leq 6 
\end{cases}
\]
1. Define defuzzification process.
2. What is the necessity to convert the fuzzy quantities into crisp quantities?
3. State the method lambda cuts employed for the conversion of the fuzzy set into crisp.
4. How is lambda cut method employed for a fuzzy relation?
5. How does the maximum method convert the fuzzy quantity to crisp quantity?
6. In what way does the Centroid method perform the defuzzification process?
7. Compare the methods employed for defuzzification process on the basis of accuracy and time consumption.
1. Two fuzzy sets $A$ and $B$ both defined on $x$ are as follows:

\[
A = \left\{ \frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.5}{x_5} + \frac{0.2}{x_6} \right\},
\]

\[
B = \left\{ \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.6}{x_5} + 0 \right\}.
\]

Find

(a) $\left( \overline{A} \right)_{0.5}$

(b) $\left( B \right)_{0.3}$

(c) $\left( A \cup \overline{A} \right)_{0.5}$

(d) $\left( A \cup B \right)_{0.4}$

(e) $\left( \overline{A} \cap B \right)_{0.6}$

(f) $\left( A \cap B \right)_{0.64}$
2. The fuzzy sets $A$, $B$, $C$ are all defined on the universe $X = [0, 5]$ with the following membership functions:

$$
\mu_A(x) = \frac{1}{1 + 5(x - 5)^2},
\mu_B(x) = 2^{-x},
\mu_C(x) = \frac{2x}{x + 5}.
$$

(a) Sketch the membership functions
(b) Define the intervals along $x$-axis corresponding to the $\lambda$ cut sets for each of the fuzzy sets $A$, $B$, $C$ for the following values of $\lambda$. $\lambda = 0.2$, $\lambda = 0.5$, $\lambda = 0.9$. 
3. For fuzzy relation $R$

\[
R = \begin{bmatrix}
0.4 & 0.3 & 0.7 & 0.5 \\
0.6 & 0.2 & 0.1 & 1 \\
0.9 & 0.8 & 0.5 & 0.6 \\
0.7 & 0.4 & 0.3 & 0.2 \\
\end{bmatrix}.
\]

Find $\lambda$ cut relations for the following values of $\lambda$

(a) $\lambda = 0.4$  (b) $\lambda = 0.7$
4. By using **maxima** and **centroid** methods, convert fuzzy output to a crisp value $z^*$ for the following graph.
5. Convert fuzzy value $z$ to precise value $z^*$ for the following graph by using **weighted average method** and **middle-of-maxima method**.

The membership functions in the above graph are symmetrical functions.